

# Boundary Effects on the Thermodynamics of Quantum Fields Near a Black Hole

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## Abstract

We study the thermodynamics of a quantum field in a spherical shell around a static black hole. We implement brick wall regularization by imposing Dirichlet boundary conditions on the field at the boundaries and analyze their effects on the free energy and the entropy. We also consider the possibility of using Neumann boundary conditions. We examine both bosonic and fermionic fields in Schwarzschild, Reissner-Nordström (RN), extreme RN, and dilatonic backgrounds. We show that the horizon divergences get contributions from the boundary (brick wall) which at the Hawking temperature are comparable to the bulk contributions. It is also shown that the leading divergence is the same for all the backgrounds considered and that the subleading logarithmic divergence is given by a specific function of surface gravity and horizon area. We will also consider the near horizon geometry and show the existence of a finite term generated by the logarithmic divergence which involves the logarithm of the horizon area. The thermodynamics of quantum fields is examined in the ultrarelativistic regime through the high temperature/small mass expansion. We derive the high temperature expansion in the presence of chemical potential by Mellin transform and heat kernel methods as applied to the harmonic sum representation of the free energy.

# 1 Introduction

The analysis of thermodynamic properties of a quantum field around a black hole is an important step towards a deeper understanding the quantum mechanical properties of black holes [1–3]. It is well known [4–15] that the free energy and the entropy of a quantum field around a black hole are plagued by divergences caused by the presence of the horizon. Therefore one needs to devise regularization methods to tackle these so-called horizon divergences. One such regularization procedure is 't Hooft's brick wall method [4] which consists of confining the quantum field inside a spherical shell around a static black hole bounded by two Dirichlet walls, *i.e.* two hypersurfaces where the quantum fields are subject to Dirichlet boundary conditions. The inner Dirichlet wall (the brick wall), situated just outside the black hole horizon, regulates the horizon divergence; and the outer wall provides an infrared cutoff. Our aim in this paper is to investigate the effects of the brick wall on the thermodynamics of both bosonic and fermionic fields in Schwarzschild, Reissner-Nordström (RN), extreme RN, dilatonic and their near horizon geometries.

Our main result is that at the Hawking temperature  $T_H$  the contribution of the brick wall to the entropy is comparable to the bulk contribution. Here the distinction between bulk and boundary contributions stems from the heat kernel expansion for the Dirichlet problem; the boundary terms are those terms in the heat kernel expansion which have their origin in the Dirichlet boundary condition on the Laplacian. It is well known that the leading bulk contribution to the entropy is proportional to the area  $A_H$  of the horizon [4, 6–15]. We will show that at the Hawking temperature the leading boundary effect is also proportional to  $A_H$  and of the same order of magnitude as the leading bulk term. More precisely we will show that the entropy of the quantum field at  $T_H$  is

$$S = B \frac{A_H}{\delta^2} + C \log \delta^2. \quad (1)$$

where

$$B = \frac{1}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) \right] + \dots, \quad (2)$$

and

$$C = C(\kappa, A_H) = \frac{\zeta(4)}{\pi^4} \left[ \frac{1}{2} - \frac{3}{2\sqrt{\pi}} \kappa A_H^{1/2} \right] - \frac{\zeta(2)}{8\pi^3} m^2 A_H + \dots \quad (3)$$

Here  $\delta$  is a cutoff parameter regulating the horizon divergences (brick wall cutoff [4]),  $\kappa$  is the surface gravity,  $A_H$  is the area of the event horizon,  $m$  is the mass parameter of the quantum field.

Both coefficients  $B$  and  $C$  will be calculated in the ultrarelativistic regime using the first four terms of the high temperature expansion in the small parameter  $\beta m$ . It will be shown that they both consist of a bulk plus a boundary contribution which are of the same order of magnitude at  $T_H$ . As is clear from the above formulae the coefficient  $B$  will be seen to be the same for all geometries considered. This is in accordance with [31] where the universality of the  $B$  coefficient was argued for by a scaling/dimensional analysis of the heat kernel coefficients in the near horizon geometry. In fact we will also examine the near horizon geometry and show that the common  $B$  coefficient has its origin in the near horizon geometry. Moreover, in the near horizon geometry we will show that the logarithmically divergent term produces a finite term in  $\log A_H$ . On the other hand in all cases considered the  $C$  coefficient will be seen to be a specific function of  $\kappa$  and  $A_H$ . We will also consider a regularization procedure involving Neumann walls and obtain analogous results for that case as well.

Instead of the original metric we will work with the optical metric [28] which is related to the former by a conformal transformation. The role of the optical metric in the computation of thermodynamic quantities in a static background was first noticed in [13] as a direct consequence of the path integral measure used in the calculation of the partition function. Here we will give an alternative derivation of the optical metric based directly on the field equations. The main technical advantage of the optical metric is that it allows us to express the single particle energies as eigenvalues of a Hamiltonian constructed therefrom. For example, for the original static metric the single particle energies arise not as eigenvalues but only as separation constants in the Klein-Gordon equation; whereas after introducing the optical metric, which is an ultra-static metric, via a conformal transformation we get the single particle energies as eigenvalues of the aforementioned Hamiltonian. This then allows us to use the heat kernel expansion to calculate the thermodynamic quantities. Here we also see the importance of the brick wall; it constraints the system in a region where the conformal transformation is free of singularities; this is the brick wall regularization procedure by which the horizon divergences are avoided.

In the following we will work in the ultrarelativistic regime where the mass parameter  $m$  of the field is much smaller than the temperature  $T$ ,  $m \ll T$ , and employ an expansion in terms of the small parameter  $\beta m$  ( $\beta = T^{-1}$ ) (sometimes called the high temperature expansion) to study the free energy and the entropy of the quantum field in the presence of Dirichlet walls. We will also present an alternative derivation [18] of the high temperature expansion based on Mellin transform methods which does not make use of the zeta function regularization. The resulting expansion is in complete agreement

with the one obtained by the latter method [19–26].

The brick wall method should not be confused with the often used volume cut-off method in which one works within the same spherical shell used in the brick wall method but without imposing any boundary conditions (see [27] for a comparison of various techniques employed in the calculation of the thermodynamic entropy of quantum fields near a black hole). In particular the heat kernel expansion used in the volume cut-off method does not involve the boundary terms. Thus in the light of our results we see that the volume cut-off method cannot be equivalent to the brick wall method which contains extra boundary terms which are of the same order of magnitude (at  $T = T_H$ ) as the bulk terms.

The leading order contribution to the entropy in the high temperature expansion was first calculated in [11] in the volume cut-off approach by a rather involved analysis based on a Liouville type quantum field theory. Later this result was reproduced in [9, 13, 17] by a simpler analysis based on the use of the optical metric. Our leading order contribution will be in complete agreement with the one given in the above papers. Higher order corrections in the high temperature expansion were obtained in the volume cut-off method by [29, 30]. The boundary effects of the brick wall on thermodynamics in the approximate near horizon geometry was first considered in [31] where the general form of the leading divergence was studied via the heat kernel coefficients. For a WKB analysis of the problem see [32–37]. The effect of boundaries in particle detection in AdS space was discussed recently in [38]. For an extensive review of the black hole entropy problem see [39].

We would like to remark that there are two different viewpoints towards the brick wall hypothesis. In [4] (and later in [16, 17]) the brick wall is a real physical barrier whose origin lies in the full theory of quantum gravity, whereas in [7] (see also [9]) the introduction of the brick wall is a regularization procedure in which the regulated horizon divergences are used to cancel the divergences in the bare gravitational constant (this is accomplished by requiring the total entropy, the entropy of the matter fields plus the entropy of the gravitons, to be finite).

Here is the brief outline of this paper. In Sec. 2 we will briefly review the optical metric construction and derive the high temperature expansion of the free energy for a neutral Bose field with non-vanishing chemical potential. We will first express the free energy as a harmonic sum and then apply the Mellin transform method together with the heat kernel expansion to get the high temperature expansion. We will also show how the resulting expansion is modified when one considers charged boson and fermions. The main results of this section are the high temperature expansions (51), (52), (55) of the free energies of neutral bosons, charged bosons and charged fermions,

respectively. In Sec. 3 we will apply the results of Sec. 2 to static black hole backgrounds such as the Schwarzschild, the RN (including the extreme limit) and the dilaton backgrounds and derive the boundary effects on the horizon divergences of the entropy, we will evaluate the latter at the Hawking temperature and compare the bulk and boundary terms with each other. We will also discuss certain issues concerning the near horizon approximation to the entropy calculation. In particular, we will use the transformation properties of the heat kernel coefficients under a conformal scaling to derive the general form of the leading divergent term in the entropy. In Sec. 4 we will summarize our results for the horizon divergences in the entropy, further analyze the logarithmic divergence, and infer the  $\kappa$  and  $A_H$  dependence of the  $C$  coefficient. In Sec. 5 we will try to understand the universality of the  $B$  coefficient by considering the near horizon geometry (this is the strategy employed in [31]). In Sec 6 we will examine the possibility of using Neumann walls as thermodynamic regulators instead of Dirichlet walls.

## 2 Thermodynamics Around the Black Hole

Consider the  $d + 1$  dimensional static metric

$$ds^2 = -F dt^2 + g_{ij} dx^i dx^j. \quad (4)$$

The free energy of the Bose gas is given by

$$\mathcal{F} = \frac{1}{\beta} \sum_{\sigma} \log(1 - e^{-\beta(\epsilon_{\sigma} - \mu)}). \quad (5)$$

Here the  $\epsilon_{\sigma}$ 's are the single particle energies determined by solving the Dirichlet problem for the Klein-Gordon equation coupled to the space-time scalar curvature  $\mathcal{R}$

$$[-\square + \xi \mathcal{R} + m^2] f_{\sigma} = 0 \quad (6)$$

by the separation of variable  $f_{\sigma}(t, x) = e^{-i\epsilon_{\sigma} t} \phi_{\sigma}(x)$  [40]. Under the scaling transformation

$$\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}, \quad \bar{f}_{\sigma} = \Omega^{\frac{1-d}{2}} f_{\sigma}, \quad (7)$$

the KG equation becomes

$$\left\{ -\square_{conf} + \frac{d-1}{4d} R + \Omega^{-2} \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) R \right] \right\} \bar{f}_{\sigma} = 0. \quad (8)$$

Here  $\square_{conf}$  and  $R$  are respectively the D'Alembertian and the scalar curvature of the transformed metric  $\bar{g}_{\mu\nu}$ . In particular if we choose  $\Omega = F^{-1/2}$  we get the optical metric

$$d\bar{s}^2 = -dt^2 + \bar{g}_{ij} dx^i dx^j, \quad (9)$$

where

$$\bar{g}_{ij} = F^{-1} g_{ij}. \quad (10)$$

Thus the KG equation in the new variables is

$$\left\{ \partial_0^2 - \Delta + \frac{d-1}{4d} R + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) \mathcal{R} \right] \right\} \bar{f}_\sigma = 0. \quad (11)$$

Here  $\Delta$  is the Laplacian of the optical metric. Since  $F$  does not depend on  $t$  we can still write the solution in the separated form  $\bar{f}_\sigma(t, x) = e^{-i\epsilon_\sigma t} \bar{\phi}_\sigma(x)$ . Thus we get the eigenvalue problem for  $\epsilon_\sigma$ 's

$$\left\{ -\Delta + \frac{d-1}{4d} R + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) \mathcal{R} \right] \right\} \bar{\phi}_\sigma = \epsilon_\sigma^2 \bar{\phi}_\sigma(x). \quad (12)$$

Defining

$$U_1 = \frac{d-1}{4d} R + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) \mathcal{R} \right], \quad (13)$$

we get

$$H_1 \bar{\phi}_\sigma = \epsilon_\sigma^2 \bar{\phi}_\sigma(x), \quad (14)$$

where

$$H_1 = -\Delta + U_1. \quad (15)$$

Now coming back to the free energy and proceeding as in [18] we write it as

$$\mathcal{F} = -\frac{1}{\beta} \sum_\sigma \sum_{k=1}^{\infty} \frac{1}{k} e^{k\beta\mu} e^{-k\beta\epsilon_\sigma} \quad (16)$$

$$= -\sum_{k=1}^{\infty} \frac{1}{k\beta} e^{k\beta\mu} \sum_\sigma e^{-k\beta\epsilon_\sigma}. \quad (17)$$

Using the subordination identity

$$e^{-b\sqrt{x}} = \frac{b}{2\sqrt{\pi}} \int_0^\infty \frac{du}{u^{3/2}} e^{-\frac{b^2}{4u}} e^{-ux}, \quad (18)$$

we get

$$\sum_\sigma e^{-k\beta\epsilon_\sigma} = \sum_\sigma e^{-k\beta m' (m'^{-1} \epsilon_\sigma)} = \frac{k\beta m'}{2\sqrt{\pi}} \int_0^\infty \frac{du}{u^{3/2}} e^{-\frac{(k\beta m')^2}{4u}} \text{Tr} e^{-um'^{-2}H}, \quad (19)$$

and

$$\mathcal{F} = - \sum_{k=1}^{\infty} e^{k\beta\mu} \frac{m'}{2\sqrt{\pi}} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{(k\beta m')^2}{4u} - u} T_r e^{-um'^{-2}H}, \quad (20)$$

Here  $m'$  is a mass parameter which will be used to form the dimensionless expansion parameter  $\beta m'$  for the high temperature expansion. Here we also defined

$$H = H_1 - m'^2 = -\Delta + U, \quad (21)$$

with

$$U = U_1 - m'^2 = \frac{d-1}{4d}R + F \left[ m^2 + \left( \xi - \frac{d-1}{4d} \right) \mathcal{R} \right] - m'^2. \quad (22)$$

Defining  $y = \beta m'$  let us write

$$\mathcal{F}(y) = -m' \sum_{k=1}^{\infty} e^{ky\bar{\mu}_r} G(ky), \quad (23)$$

with

$$G(y) = \frac{e^{y\bar{\epsilon}_0}}{2\sqrt{\pi}} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{y^2}{4u} - u} T_r e^{-um'^{-2}H}. \quad (24)$$

and

$$\mu_r = \mu - \epsilon_0, \quad \bar{\mu} = \frac{\mu}{m'}, \quad \bar{\mu}_r = \frac{\mu_r}{m'}, \quad \bar{\epsilon}_0 = \frac{\epsilon_0}{m'}. \quad (25)$$

Now the sum in (23) is a harmonic sum which is by definition a series of the form

$$f(y) = \sum_{k=1}^{\infty} F(ky) \quad (26)$$

The small  $y = \beta m$  asymptotic of this harmonic sum is completely determined by the poles and the residues of the meromorphic extension of the Mellin transform of  $\mathcal{F}$  [41]. Recall that the Mellin transform of a function  $f(y)$  is given by

$$(\mathcal{M}f(y))(s) = \tilde{f}(s) = \int_0^{\infty} dy y^{s-1} f(y). \quad (27)$$

Let us also recall the definition of the Laplace-Mellin transform [42] of  $f$

$$(\mathcal{LM}f(y))(s, z) = \tilde{f}(s, z) = \int_0^{\infty} dy y^{s-1} e^{-zs} f(y). \quad (28)$$

The asymptotic expansion of  $f(y)$  is then given according to the following rule [41]. If the Mellin transform has the singular expansion ( $\asymp$  means the singular part of the expansion)

$$\tilde{f}(s) \asymp \sum_{w,k} \frac{A(w,k)}{(s-w)^{k+1}}, \quad (29)$$

then

$$f(y) \sim \sum_{w,k} A(w,k) \frac{(-1)^k}{k!} y^{-w} (\log y)^k. \quad (30)$$

If  $f(y)$  is given by a harmonic sum (26) then it is easy to see that

$$\tilde{f}(s) = \zeta(s) \tilde{F}(s). \quad (31)$$

Now we can apply this method to  $\mathcal{F}(y)$ . The Mellin transform is given by

$$\mathcal{M}(-\beta \mathcal{F}(y))(s) = \zeta(s) \mathcal{M}(e^{y\bar{\mu}_r} G(y))(s). \quad (32)$$

But using the definition of the Laplace-Mellin transform we can also write this as

$$\mathcal{M}(-\beta \mathcal{F}(y))(s) = \zeta(s) \tilde{G}(s, -\mu). \quad (33)$$

Here

$$\tilde{G}(s, -\mu) = \int_0^\infty dy y^{s-1} e^{\bar{\mu}_r s} G(y). \quad (34)$$

Now the singularities of this integral arise from the  $y \rightarrow 0$  limit. First let us use the heat kernel expansion

$$Tr e^{-uH} = \frac{1}{(4\pi u)^{d/2}} (a_0 + a_{1/2} u^{1/2} + a_1 u + \dots) \quad (35)$$

to get

$$\int_0^\infty \frac{du}{u^{3/2}} e^{-\frac{y^2}{4u}-u} Tr e^{-um'^{-2}H} = \sum_{n=0}^\infty 2m'^{d-n} a_{n/2} \left(\frac{y}{2}\right)^{\frac{n-d-1}{2}} K_{\frac{-n+d+1}{2}}(y). \quad (36)$$

Here  $K$ 's are the modified Bessel functions of second kind. Using their series expansions we arrive at

$$\frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{du}{u^{3/2}} e^{-\frac{y^2}{4u}-u} Tr e^{-um'^{-2}h} \asymp b_{-d-1} y^{-d-1} + b_{-d} y^{-d} + \dots \quad (37)$$



where  $b$  coefficients are

$$\begin{aligned}
b_{-d-1} &= \frac{(2m')^d}{\sqrt{\pi}} \Gamma\left(\frac{d+1}{2}\right) a_0, \\
b_{-d} &= \frac{(2m')^{d-1}}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) a_{1/2}, \\
b_{-d+1} &= \frac{(2m')^{d-2}}{\sqrt{\pi}} \Gamma\left(\frac{d-1}{2}\right) (-m'^2 a_0 + a_1)
\end{aligned} \tag{38}$$

Here we listed only the first few coefficients that we will need in the following. Also note that for the Dirichlet problem the heat kernel coefficients for the operator  $-\Delta + U$  are [43, 44]

$$\begin{aligned}
a_0 &= \frac{1}{(4\pi)^{d/2}} \int_B dV, \\
a_{1/2} &= -\frac{1}{4(4\pi)^{\frac{d-1}{2}}} \int_{\partial B} dS, \\
a_1 &= \frac{1}{6(4\pi)^{d/2}} \left[ \int_B dV (-6U + R) + 2 \int_{\partial B} dS K_{aa} \right], \\
a_{3/2} &= -\frac{1}{4} \frac{1}{(4\pi)^{(d-1)/2}} \frac{1}{96} \int_{\partial B} dS [16(-6U + R) + 8R_{aNaN} + \\
&\quad + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}].
\end{aligned} \tag{39}$$

Here one uses an orthonormal basis  $E_a, N$  adapted to  $\partial B$  where  $E_a$ 's are tangent to  $\partial B$  and  $N$  is the inward normal to it. Moreover,  $R_{ijkl}$  is the Riemann tensor,  $K_{ab}$  is the extrinsic curvature of  $\partial B$  and summation over repeated orthonormal basis indices  $a, b$  is assumed.

Now plugging (36) into (24), expanding the exponential pre-factor, and multiplying the series we get for small  $y$ ;

$$G(y) \asymp c_{-d-1} y^{-d-1} + c_{-d} y^{-d} + c_{-d+1} y^{-d+1} + \dots, \tag{40}$$

where the coefficients of interest to us are

$$\begin{aligned}
c_{-d-1} &= b_{-d-1}, \quad c_{-d} = b_{-d} + \bar{\epsilon}_0 b_{-d-1} \\
c_{-d+1} &= b_{-d+1} + \bar{\epsilon}_0 b_{-d} + \frac{1}{2} \bar{\epsilon}_0^2 b_{-d-1}.
\end{aligned} \tag{41}$$

So

$$\begin{aligned}
\tilde{G}(s, -\mu) &= \int_0^\infty dy y^{s-1} e^{\bar{\mu}_r y} G(y) \\
&\sim c_{-d-1} \frac{\Gamma(s-d-1)}{(-\bar{\mu}_r)^{s-d-1}} + c_{-d} \frac{\Gamma(s-d)}{(-\bar{\mu}_r)^{s-d}} + \\
&\quad + c_{-d+1} \frac{\Gamma(s-d+1)}{(-\bar{\mu}_r)^{s-d+1}} + \dots
\end{aligned} \tag{42}$$

Around  $-m$  ( $m = 0, 1, 2, \dots$ ),

$$\Gamma(s-n) \sim \frac{(-1)^m}{m!} \frac{1}{s-n+m}. \tag{43}$$

and we also have the simple pole of  $\zeta$  at  $s = 1$

$$\zeta(s) \sim \frac{1}{s-1} + \gamma. \tag{44}$$

Here  $\gamma$  is the Euler-Mascheroni constant. Thus the poles of  $\zeta(s)\tilde{G}(s, -\mu)$  are seen to be the integers  $d, d-1, d-2, \dots$ . Here all the poles are simple except the one at  $s = 1$  which is a double pole. Thus we have the following residues:

Around  $s = d+1$ :

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d+1)c_{-d-1} \frac{1}{s-d-1}. \tag{45}$$

around  $s = d$ :

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d)(c_{-d-1}\bar{\mu}_r + c_{-d}) \frac{1}{s-d}, \tag{46}$$

around  $s = d-1$ :

$$\zeta(s)\tilde{G}(s, -\mu) \asymp \zeta(d-1) \left( \frac{1}{2}c_{-d-1}\bar{\mu}_r^2 + c_{-d}\bar{\mu}_r + c_{-d+1} \right) \frac{1}{s-d+1}. \tag{47}$$

However, for  $d = 3$ ,  $d-2 = 1$  and we have a double pole at  $s = 1$ . Thus for  $d = 3$  around  $s = 1$  we have

$$\begin{aligned}
&\zeta(s)\tilde{G}(s, -\mu) \asymp \left( \frac{1}{2}c_{-d}\mu_r^2 + c_{-d+1}\mu_r + c_{-d+2} \right) \frac{1}{(s-d+2)^2} \\
&+ \left[ FP\tilde{G}(s=1, -\mu) + \gamma \left( \frac{1}{2}c_{-d}\mu_r^2 + c_{-d+1}\mu_r + c_{-d+2} \right) \right] \frac{1}{s-d+2}.
\end{aligned} \tag{48}$$

Here  $FP$  means the finite part of the integral

So the small  $y$  asymptotic of  $\mathcal{F}$  is

$$\begin{aligned} \mathcal{F} \sim & -m'\zeta(d+1)c_{-d-1} \left(\frac{T}{m'}\right)^{d+1} - m'\zeta(d)(c_{-d-1}\bar{\mu}_r + c_{-d}) \left(\frac{T}{m'}\right)^d \\ & - m'\zeta(d-1) \left(\frac{1}{2}c_{-d-1}\bar{\mu}_r^2 + c_{-d}\bar{\mu}_r + c_{-d+1}\right) \left(\frac{T}{m'}\right)^{d-1} + \dots \quad (49) \end{aligned}$$

Here using (41) we get

$$\begin{aligned} c_{-d-1} &= b_{-d-1} = \frac{(2m')^d}{\sqrt{\pi}} \Gamma\left(\frac{d+1}{2}\right) a_0 \\ c_{-d-1}\bar{\mu}_r + c_{-d} &= \bar{\mu}b_{-d-1} + b_{-d} = \bar{\mu} \frac{(2m')^d}{\sqrt{\pi}} \Gamma\left(\frac{d+1}{2}\right) a_0 + \frac{(2m')^{d-1}}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) a_{1/2} \\ \frac{1}{2}c_{-d-1}\bar{\mu}_r^2 + c_{-d}\bar{\mu}_r + c_{-d+1} &= \frac{1}{2}\bar{\mu}^2 b_{-d-1} + \bar{\mu}b_{-d} + b_{-d+1} \\ &= \frac{1}{2}\bar{\mu}^2 \frac{(2m')^d}{\sqrt{\pi}} \Gamma\left(\frac{d+1}{2}\right) a_0 + \bar{\mu} \frac{(2m')^{d-1}}{\sqrt{\pi}} \Gamma\left(\frac{d}{2}\right) a_{1/2} \\ &+ \frac{(2m')^{d-2}}{\sqrt{\pi}} \Gamma\left(\frac{d-1}{2}\right) (-m'^2 a_0 + a_1). \quad (50) \end{aligned}$$

Specializing to  $d = 3$  we get

$$\begin{aligned} \mathcal{F} &= -\frac{\zeta(4)}{\sqrt{\pi}} 8a_0 T^4 - \frac{\zeta(3)}{\sqrt{\pi}} [8\mu a_0 + 2\sqrt{\pi} a_{1/2}] T^3 \\ &- \frac{\zeta(2)}{\sqrt{\pi}} [(4\mu^2 - 2m'^2)a_0 + 2\sqrt{\pi}\mu a_{1/2} + 2a_1] T^2 \\ &+ \frac{1}{2\sqrt{\pi}} \left[ \left(\frac{8}{3}\mu^3 - 4m'^2\mu\right) a_0 + 2\sqrt{\pi}(\mu^2 - m'^2)a_{1/2} + 4\mu a_1 + 2\sqrt{\pi} a_{3/2} \right] \\ &\times T \log(T/m') \\ &- \frac{1}{2\sqrt{\pi}} \left( \gamma \left[ \left(\frac{8}{3}\mu^3 - 4m'^2\mu\right) a_0 + 2\sqrt{\pi}(\mu^2 - m'^2)a_{1/2} + 4\mu a_1 + 2\sqrt{\pi} a_{3/2} \right] \right. \\ &+ \left. FP(\mathcal{LMG}(s=1, |\mu|)) T + \dots \right] \quad (51) \end{aligned}$$

In the case of charged bosons we have

$$\begin{aligned}
\mathcal{F}_{charged} &= \mathcal{F}(\mu) + \mathcal{F}(-\mu) = -\frac{\zeta(4)}{\sqrt{\pi}} 16 a_0 T^4 - \frac{\zeta(3)}{\sqrt{\pi}} 4 a_{1/2} T^3 \\
&\quad - \frac{\zeta(2)}{\sqrt{\pi}} 4 [(2\mu^2 - m'^2) a_0 + a_1] T^2 \\
&\quad + \left[ 2(\mu^2 - m'^2) a_{1/2} + \frac{2}{\pi} a_{3/2} \right] \times T \log(T/m') \\
&\quad - \left( \gamma [2(\mu^2 - m'^2) a_{1/2} + 2 a_{3/2}] + \frac{1}{\sqrt{\pi}} FP(\mathcal{LMG})(1, |\mu|) \right) T \\
&\quad + \dots
\end{aligned} \tag{52}$$

Finally, for a Fermi gas we have

$$\mathcal{F}_{fermion} = -\frac{2}{\beta} \sum_{\sigma} [\log(1 + e^{-\beta(\epsilon_{\sigma} - \mu)}) + \log(1 + e^{-\beta(\epsilon_{\sigma} + \mu)})] \tag{53}$$

Again expanding the logarithms we get

$$\mathcal{F} = -4 \sum_{\sigma} \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k\beta} \cosh(k\beta\mu) e^{-k\beta\epsilon_{\sigma}} \tag{54}$$

The rest of the analysis parallels the bosonic case. Because of the alternating nature of the series expansion the Mellin transform now generates the Dirichlet  $\eta$  function instead of the Riemann  $\zeta$ .

$$\begin{aligned}
\mathcal{F}_{fermion} &= -2 \frac{\eta(4)}{\sqrt{\pi}} 16 a_0 T^4 - 2 \frac{\eta(3)}{\sqrt{\pi}} 4 a_{1/2} T^3 \\
&\quad - 2 \frac{\eta(2)}{\sqrt{\pi}} 4 [(2\mu^2 - m'^2) a_0 + a_1] T^2 \\
&\quad - 2\eta(1) \left[ 2(\mu^2 - m'^2) a_{1/2} + \frac{2}{\pi} a_{3/2} \right] T + \dots
\end{aligned} \tag{55}$$

Here

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}. \tag{56}$$

Unlike the bosonic case we do not have a logarithmic term in this expansion since the  $\eta$  function is entire and consequently the poles of the Mellin transform are all simple.

If we now go back to (19), the very first place we introduced the parameter  $m'$ , we see that the resulting expansion should be independent of  $m'$ . In the next section we will show, by direct calculation, that this is indeed the case; for example we will see that in the term  $(2\mu^2 - m'^2)a_0 + a_1$ , the  $m'^2$  terms (one explicit in  $-m'^2 a_0$  and one hidden in  $a_1$ ) do cancel each other. The only place where the cancelation is not obvious is the logarithmic term  $\log(T/m')$  in the bosonic case. However, on general grounds we expect that this logarithmic term will be canceled by a similar term coming from  $FP(\mathcal{LMG})(1, |\mu|)T$  appearing in the next order of the expansion. A more detailed analysis of this regularised finite part integral will be given elsewhere.

### 3 Boundary Effects in Black Hole Backgrounds

We will consider Schwarzschild, Reissner-Nordström (RN) and dilaton black holes [45, 46] in 1+3 dimensions and calculate the free energy and the entropy for a neutral Bose field. For charged bosons and charged fermions the results can be written down at once with the minor modifications explained at the end of Sec 2. We will focus on the case of vanishing chemical potential.

So setting  $\mu = 0$  and using the explicit forms of the heat kernel coefficients (39) in (51) we get

$$\begin{aligned}
\mathcal{F} = & -\zeta(4) \frac{1}{\pi^2} VT^4 + \zeta(3) \left( \frac{1}{8\pi} A \right) T^3 \\
& - \frac{\zeta(2)}{4\pi^2} \left[ \int_B dV (-F) \left[ m^2 + \left( \xi - \frac{1}{6} \right) \mathcal{R} \right] + \int_{\partial B} dS \frac{K_{aa}}{3} \right] T^2 \\
& - \frac{1}{16\pi} \int_{\partial B} dS \left\{ -F \left[ m^2 + \left( \xi - \frac{1}{6} \right) \mathcal{R} \right] + \right. \\
& \left. + \frac{1}{96} [8R_{NaNa} + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}] \right\} T \log \frac{T}{m'} \\
& + \dots
\end{aligned} \tag{57}$$

and the entropy is given as

$$\begin{aligned}
S = & -\frac{\partial \mathcal{F}}{\partial T} = 4\zeta(4)\frac{1}{\pi^2}VT^3 - 3\zeta(3)\left(\frac{1}{8\pi}A\right)T^2 \\
& + \frac{\zeta(2)}{2\pi^2}\left[\int_B dV(-F)\left[m^2 + \left(\xi - \frac{1}{6}\right)\mathcal{R}\right] + \int_{\partial B} dS\frac{K_{aa}}{3}\right]T \\
& + \frac{1}{16\pi}\int_{\partial B} dS\left\{-F\left[m^2 + \left(\xi - \frac{1}{6}\right)\mathcal{R}\right] + \right. \\
& \left. + \frac{1}{96}[8R_{NaNa} + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}]\right\}\log\frac{T}{m'} \\
& + \dots
\end{aligned} \tag{58}$$

As promised we see that the  $m'$  terms disappear, except in the logarithmic term.

In all geometries we consider here the metric has the general form

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r(r-a)[d\theta^2 + \sin^2\theta d\phi^2], \tag{59}$$

where  $a = 0$  for Schwarzschild and RN space-times. We will take  $B$  as the "spherical" shell defined by  $r_1 \leq r \leq r_2$  with  $r_1 = r_H + \epsilon$ , where  $r_H$  is the radial coordinate of the horizon and  $\epsilon$  is the brick wall cutoff.

The optical metric is

$$d\bar{s}^2 = F^{-2}(r)dr^2 + r(r-a)F^{-1}(r)[d\theta^2 + \sin^2\theta d\phi^2]. \tag{60}$$

The quantities needed for the calculation of the heat kernel coefficients (39) are then as follows.

The volume element is given as

$$dV = F^{-2}r(r-a)\sin\theta dr \wedge d\theta \wedge d\phi. \tag{61}$$

The area element of a constant  $r$  surface is given as

$$dS_r = F^{-1}r(r-a)\sin\theta d\theta \wedge d\phi, \tag{62}$$

The inward normal to the inner wall is given by the value at  $r = r_1$  of the vector field

$$N = F\partial_r, \tag{63}$$

and the trace of the corresponding intrinsic curvature can be calculated as

$$K_{aa} = -\nabla \cdot N = F'(r) - \frac{(2r-a)F(r)}{r(r-a)}. \tag{64}$$

Consider the orthonormal basis

$$N_{(r)} = F\partial_r, \quad E_1 = \frac{\sqrt{F}}{\sqrt{r(r-a)}}\partial_\theta, \quad E_2 = \frac{\sqrt{F}}{\sqrt{r(r-a)}\sin\theta}\partial_\phi. \quad (65)$$

This basis is adapted to  $r = \text{const.}$  surface, that is  $N$  is normal to that surface and  $T$ 's are tangent to it. Moreover on the inner wall  $r = r_1$  of  $B$ ,  $N$  is the inward normal of  $\partial B$ .

The extrinsic curvature of the inner wall at  $r = r_1$  is given by

$$\begin{aligned} K_{ab} &= N \cdot (\nabla_{E_a} E_b) \\ &= N_\nu E_a^\mu \partial_\mu E_b^\nu + \Gamma_{\mu\sigma}^\nu E_a^\mu E_b^\sigma N_\nu \\ &= \Gamma_{\mu\sigma}^\nu E_a^\mu E_b^\sigma N_\nu. \end{aligned} \quad (66)$$

More explicitly, using  $N_\nu = F^{-1}\delta_\nu^0$ ,

$$K_{11} = \Gamma_{\theta\theta}^r \frac{1}{r(r-a)}, \quad K_{22} = \Gamma_{\phi\phi}^r \frac{1}{r(r-a)\sin^2\theta}, \quad K_{12} = K_{21} = \Gamma_{\theta\phi}^r \frac{1}{r(r-a)\sin\theta}. \quad (67)$$

Similarly,

$$R_{aNaN} = \frac{F^3}{r(r-a)} R_{\theta r \theta r} + \frac{F^3}{r(r-a)\sin^2\theta} R_{\phi r \phi r}. \quad (68)$$

### 3.1 Schwarzschild Black Hole

We will now specialize to 1 + 3 Schwarzschild geometry for which

$$F = 1 - \frac{2M}{r}, \quad \mathcal{R} = 0, \quad (69)$$

and the extrinsic curvature is

$$K_{11} = K_{22} = \frac{3M-r}{r^2}, \quad K_{12} = 0, \quad (70)$$

which implies

$$K_{aa} = 2\frac{3M-r}{r^2}, \quad K_{ab}K_{ab} = 2\frac{(3M-r)^2}{r^4}. \quad (71)$$

And finally,

$$R_{aNaN} = \frac{2M(3M-2r)}{r^4}. \quad (72)$$

Now using (61) we get the volume of  $B$  as

$$V = 4\pi \left[ \frac{r^3}{3} + 2Mr^2 + 12M^2r + 32M^3 \log(r - 2M) - \frac{16M^4}{r - 2M} \right]_{r_1}^{r_2}. \quad (73)$$

Therefore

$$V \asymp 64\pi M^3 \left( \frac{M}{\epsilon} - 2 \log(\epsilon) \right). \quad (74)$$

Similarly using (62) the area of  $\partial B$  is given as

$$A = 4\pi \left( \frac{r_1^3}{r_1 - 2M} + \frac{r_2^3}{r_2 - 2M} \right). \quad (75)$$

So  $A$  diverges as

$$A \asymp \frac{32\pi M^3}{\epsilon}. \quad (76)$$

At  $O(T)$  the bulk contribution is

$$\begin{aligned} \int_B dV (-F) m^2 &= -4\pi m^2 \int_{r_1}^{r_2} dr \frac{r^2}{F} \\ &= -4\pi m^2 \left[ (4M^2r + Mr^2 + \frac{r^3}{3} + 8M^3 \log(r - 2M)) \right]_{r_1}^{r_2} \\ &\asymp 32\pi M^3 m^2 \log \epsilon. \end{aligned} \quad (77)$$

At the same order the boundary contribution to the horizon divergence comes only from the integral over  $r = r_1$  and is seen to be

$$\begin{aligned} \int_{r=r_1} dS_{r_1} \frac{1}{3} K_{aa} &= \frac{8\pi}{3} \left[ \frac{r_1(3M - r_1)}{r_1 - 2M} \right] \\ &\asymp \frac{16\pi M^2}{3} \frac{1}{\epsilon} \end{aligned} \quad (78)$$

and

$$\begin{aligned} &\frac{1}{16\pi} \int_{\partial B} dS \left\{ -F m^2 + \frac{1}{96} [8R_{NaNa} + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}] \right\} \\ &= \frac{1}{48} \left[ \frac{2M(3M - 2r) + (3M - r)^2}{r(r - 2M)} \right] \\ &\asymp -\frac{M}{96} \frac{1}{\epsilon} \end{aligned} \quad (79)$$



If we compare various bulk and boundary terms appearing in the coefficients of the expansion (58) we get

$$\frac{A}{V} = O(M^{-1}), \quad (80)$$

$$\frac{\int_{\partial B} dS \frac{1}{3} K_a a}{V} = O(M^{-2}), \quad (81)$$

$$\frac{\int_B F m^2}{V} \rightarrow 0. \quad (82)$$

Thus for generic values of  $T$  we see that the boundary contributions are suppressed against the bulk contributions by inverse powers of  $M$ . However we must evaluate the entropy at the Hawking temperature  $T_H$  which is  $O(M^{-1})$ . In this case we have for example

$$\frac{AT_H^2}{VT_H^3} = O(M^0), \quad (83)$$

and the boundary contributions become as important as the bulk terms.

At this point we evaluate the entropy (58) at the Hawking temperature

$$T_H = \frac{1}{8\pi M} = \frac{\kappa}{2\pi}. \quad (84)$$

where  $\kappa$  is the surface gravity. We also trade the cutoff  $\epsilon$  with the proper length cutoff  $\delta$

$$\epsilon = T_H \pi \delta^2 = \frac{\delta^2}{8M}. \quad (85)$$

The result is

$$\begin{aligned} S &= \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) - \frac{\pi^2}{48} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}. \end{aligned} \quad (86)$$

Here  $L$  is an infrared cutoff which may be taken as the radial coordinate of the outer wall of  $B$ . At this point we can use  $m'$  independence of our expansion to set  $m' = T_H$  and thus get rid of the logarithmic term. However we prefer to leave the above expression as it is in order to point out the universal features of the  $O(\delta^2)$  divergence. The terms proportional to  $\zeta(4)$  originate from the leading  $O(T^3)$  term of the high temperature expansion which is proportional to the optical volume and are in complete agreement with the results of [11] and [13]. Moreover the dependence of  $S$  on the cutoff  $\delta$  is of the right form for the cancelation against the divergences in the reciprocal of the bare gravitational constant [5, 7].

### 3.2 Reissner-Nordst rm Black Hole

For the RN background we have

$$F(r) = \left(1 - \frac{r_-}{r}\right) \left(1 - \frac{r_+}{r}\right), \quad \mathcal{R} = 0, \quad (87)$$

$$K_{11} = K_{22} = \frac{3r(r_+ + r_-) - 2r^2 - 4r_+r_-}{2r^3}, \quad K_{12} = 0, \quad (88)$$

which implies

$$K_{aa} = \frac{-2r^2 + 3(r_+ + r_-)r - 4r_+r_-}{r^3}, \quad K_{ab}K_{ab} = 2 \left[ \frac{3r(r_+ + r_-) - 2r^2 - 4r_+r_-}{2r^3} \right]^2. \quad (89)$$

$$R_{aNaN} = \frac{8r_+^2r_-^2 - 12r_+r_-(r_+ + r_-)r + 3(r_+^2 + 6r_+r_- + r_-^2)r^2 - 4(r_+ + r_-)r^3}{2r^6}. \quad (90)$$

Thus using (61) we get the volume of the spherical shell as

$$\begin{aligned} V = & 4\pi \left[ \frac{r_-^6}{(r_- - r_+)^2(r_- - r)} + \frac{r_+^6}{(r_- - r_+)^2(r_+ - r)} + (3r_-^2 + 4r_-r_+ + 3r_+^2)r \right. \\ & + (r_- + r_+)r^2 + \frac{r^3}{3} + \frac{2r_-^5(2r_- - 3r_+)\log(r - r_-)}{(r_- - r_+)^3} \\ & \left. + \frac{2(3r_- - 2r_+)r_+^5\log(r - r_+)}{(r_- - r_+)^3} \right]. \end{aligned} \quad (91)$$

thus

$$V \asymp 4\pi \frac{r_+^5}{(r_- - r_+)^2} \left[ \frac{r_+}{\epsilon} + \frac{3r_- - 2r_+}{r_+ - r_-} 2\log \epsilon \right]. \quad (92)$$

Similarly using (62) we get

$$A \asymp \frac{4\pi r_+^4}{r_+ - r_-} \frac{1}{\epsilon}. \quad (93)$$

On the other hand

$$\begin{aligned}
\int_B (-F) m^2 &= -4\pi m^2 \int_{r_1}^{r_2} dr \frac{r^2}{F} \\
&= -4\pi m^2 \left[ (r_+^2 + r_-^2 + r_+ r_-) r + \frac{1}{2} (r_+ + r_-) r^2 + \frac{r^3}{3} \right. \\
&\quad \left. + \frac{r_-^4}{r_+ - r_-} \log(r - r_-) + \frac{r_+^4}{r_+ - r_-} \log(r - r_+) \right]_{r_1}^{r_2} \\
&\asymp \frac{4\pi m^2 r_+^4}{r_+ - r_-} \log \epsilon.
\end{aligned} \tag{94}$$

Finally, using (64), the surface integral of extrinsic curvature over the inner wall of  $B$  is

$$\int_{r=r_1} dS_r K_{aa} = -4\pi \frac{2r_1^3 - 3r_1^2(r_+ + r_-) + 4r_1 r_+ r_-}{(r_1 - r_-)(r_1 - r_+)}, \tag{95}$$

Thus

$$\int_{\partial B} dS \frac{1}{3} K_{aa} \asymp \frac{4\pi}{3} \frac{r_+^2}{\epsilon}. \tag{96}$$

and

$$\begin{aligned}
&\frac{1}{16\pi} \int_{\partial B} dS \left\{ -F m^2 + \frac{1}{96} [8R_{NaNa} + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}] \right\} \\
&\asymp -\frac{(r_+ - r_-)}{192} \frac{1}{\epsilon}
\end{aligned} \tag{97}$$

Now evaluating  $S$  at the Hawking temperature

$$T_H = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\kappa}{2\pi} \tag{98}$$

where  $\kappa$  is the surface gravity.

$$A_H = 4\pi r_+^2 \tag{99}$$

with

$$\epsilon = T_H \pi \delta^2 = \frac{(r_+ - r_-)}{4r_+^2} \delta^2, \tag{100}$$

we get

$$\begin{aligned}
S &= \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) - \frac{\pi^2}{48} \log \frac{\kappa}{2\pi m} \right] \frac{1}{\delta^2} \\
&+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{2r_+ - 3r_-}{2r_+} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \left( \frac{\delta^2}{L^2} \right).
\end{aligned} \tag{101}$$

For the extreme Reissner-Nordst rm Black Hole case i.e.  $\lim r_- \rightarrow r_+$  we have

$$\begin{aligned}
S &= \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) - \frac{\pi^2}{48} \log \frac{\kappa}{2\pi m} \right] \frac{1}{\delta^2} \\
&+ \frac{1}{\pi^2} \left[ \frac{\zeta(4)}{\pi^2} \frac{1}{2} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \left( \frac{\delta^2}{L^2} \right). \\
&+ \dots
\end{aligned} \tag{102}$$

Note that the coefficient of the  $\delta^{-2}$  term is the same as in the Schwarzschild case.

### 3.3 Dilaton Black Hole

For the dilaton metric

$$F(r) = \left( 1 - \frac{2M}{r} \right), \quad \mathcal{R} = \frac{a^2(r - 2M)}{2(r - a)^2 r^3}, \tag{103}$$

$$K_{11} = K_{22} = \frac{(-4aM + ar + 6Mr - 2r^2)}{2r^2(r - a)}, \quad K_{12} = 0, \tag{104}$$

which implies

$$K_{aa} = \frac{-2r^2 + (6M + a)r - 4aM}{r^2(r - a)}, \quad K_{ab}K_{ab} = 2 \left[ \frac{(-4aM + ar + 6Mr - 2r^2)}{2r^2(r - a)} \right]^2. \tag{105}$$

$$R_{aNaN} = \frac{4M(3M - 2r)r^2 + 8aMr(-3M + 2r) + a^2(16M^2 - 12Mr + r^2)}{2(a - r)^2 r^4}. \tag{106}$$

and consequently we have

$$V \asymp 4\pi \left( 8M^3(2M - a)\frac{1}{\epsilon} + 4M^2(3a - 8M) \log \epsilon \right) \quad (107)$$

$$A \asymp 4\pi \left( 4M^2(2M - a)\frac{1}{\epsilon} \right) \quad (108)$$

$$\int_B (-F) \left[ m^2 + \left( \xi - \frac{1}{6} \right) R \right] \asymp \frac{8\pi}{3} m^2 M^2 (2M - a) \log \epsilon \quad (109)$$

Integrating over the inner wall of  $B$  gives

$$\int_{\partial B} dS \frac{1}{3} K_{aa} \asymp \frac{4\pi}{3} \left( 2M(2M - a)\frac{1}{\epsilon} \right) \quad (110)$$

and

$$\begin{aligned} & \frac{1}{16\pi} \int_{\partial B} dS \left\{ -F \left[ m^2 + \left( \xi - \frac{1}{6} \right) R \right] + \right. \\ & \left. + \frac{1}{96} [8R_{NaNa} + 7K_{aa}K_{bb} - 10K_{ab}K_{ab}] \right\} \\ & \asymp -\frac{(a - 2M)}{192} \frac{1}{\epsilon} \end{aligned} \quad (111)$$

After setting  $A_H = 4\pi 2M(2M - a)$ ,  $T_H = 1/(8\pi M) = \kappa/2\pi$  for the entropy we have;

$$\begin{aligned} S &= \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) - \frac{\pi^2}{48} \log \frac{\kappa}{2\pi m} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{(8M - 3a)}{8M} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}. \end{aligned} \quad (112)$$

Note that the coefficient of the  $\delta^{-2}$  term is the same as in the Schwarzschild and RN cases.

## 4 Horizon Divergences

Now summarizing our results we have

$$S = BA_H \frac{1}{\delta^2} + C \log \delta^2. \quad (113)$$

In all cases considered  $B$  coefficient has the same value

$$B = \frac{1}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) \right]. \quad (114)$$

Here, as explained at the end of Sec. 2, we dropped the logarithmic term by choosing  $m' = \kappa/2\pi$ . On the other hand the  $C$  coefficients are all different from each other. However, we will now show that if the  $C$  coefficients are expressed in terms of the surface gravity  $\kappa$  and the horizon area  $A_H$  they all represent the same function  $C(\kappa, A_H)$ .

Let us start with the RN space-time for which

$$C_{RN} = \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{2r_+ - 3r_-}{2r_+} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \quad (115)$$

Now for RN we have  $\kappa = (r_+ - r_-)/4\pi r_+^2$ ,  $A_H = 4\pi r_+^2$ . So we have  $r_+ - r_- = \kappa A_H$  and  $r_+ = \sqrt{A_H}/2\sqrt{\pi}$ . Using these we can express  $C_{RN}$  in terms of  $\kappa$  and  $A_H$  as

$$C_{RN} = \frac{\zeta(4)}{\pi^4} \left[ \frac{1}{2} + \frac{3}{2\sqrt{\pi}} \kappa \sqrt{A_H} \right] - \frac{\zeta(2)}{8\pi^3} m^2 A_H \quad (116)$$

Now let us define the function  $C(\kappa, A_H)$  by the expression on the right hand side of this equation.

For the Schwarzschild metric we have  $\kappa = 1/4M$  and  $A_H = 16\pi M^2$ . So we see that

$$C\left(\frac{1}{4M}, 16\pi M^2\right) = -\frac{\zeta(4)}{\pi^4} + \frac{\zeta(2)}{8\pi^3} m^2 A_H = C_{Schwarzschild}. \quad (117)$$

For extreme RN we have  $\kappa = 0$  and  $A_H = 4\pi r_+^2$ . Thus we get

$$C(0, 4\pi r_+^2) = \frac{\zeta(4)}{2\pi^4} + \frac{\zeta(2)}{8\pi^3} m^2 A_H = C_{extremeRN}. \quad (118)$$

For the dilaton background we have a slight complication due to the dilaton charge  $a$ . Now we have  $\kappa = 1/4M$ ,  $A_H = 4\pi r_+(r_+ - a)$  and using these we can write

$$\begin{aligned} C_{dilaton} &= \frac{\zeta(4)}{2\pi^4} \left[ -\frac{8M - 3a}{2M} \right] + \frac{\zeta(2)}{8\pi^3} m^2 A_H \\ &= \frac{\zeta(4)}{2\pi^4} \left[ \frac{1}{2} - \frac{3}{2\sqrt{\pi}} \kappa A_H \frac{1 - a\kappa}{\sqrt{1 - 2a\kappa}} \right] + \frac{\zeta(2)}{8\pi^3} m^2 A_H. \end{aligned} \quad (119)$$

Thus we arrive at the generalization

$$C(\kappa, A_H, a) = \frac{\zeta(4)}{2\pi^4} \left[ \frac{1}{2} - \frac{3}{2\sqrt{\pi}} \kappa A_H \frac{1 - a\kappa}{\sqrt{1 - 2a\kappa}} \right] - \frac{\zeta(2)}{8\pi^3} m^2 A_H, \quad (120)$$

which reduces to  $C(\kappa, A_H)$  when  $a = 0$ .

## 5 Near Horizon Approximation

The near horizon approximation to the metric

$$ds^2 = -F(r)dt^2 + F^{-1}(r)dr^2 + r(r-a)(d\theta^2 + \sin^2\theta d\phi^2) \quad (121)$$

is given as

$$ds_{NH}^2 = -2\kappa\rho' dt^2 + (2\kappa\rho')^{-1} d\rho'^2 + r_+(r_+ - a)(d\theta^2 + \sin^2\theta d\phi^2). \quad (122)$$

Here  $r_+$  is the radial coordinate of the event horizon and  $\kappa = 2F'(r_+)$  is the surface gravity thereof. The brick wall is situated at  $\rho' = \epsilon$ . The proper radial distance of the brick wall to the horizon is given by

$$\delta = \int_0^\epsilon \frac{d\rho'}{\sqrt{2\kappa\rho'}} = \sqrt{\frac{2\epsilon}{\kappa}}. \quad (123)$$

This is clearly the proper length cut-off we have been using throughout the paper.

The scalar curvature of the near horizon metric is

$$\mathcal{R} = \frac{2}{r_+(r_+ - a)} = \frac{8\pi}{A_H}. \quad (124)$$

Here  $A_H = 4\pi r_+(r_+ - a)$  is the area of the event horizon.

The corresponding optical metric is

$$d\tilde{s}_{NH}^2 = \frac{4}{\kappa^2} \left[ \frac{d\rho'^2}{\rho'^2} + \frac{2\kappa r_+(r_+ - a)}{\rho'} (d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (125)$$

Upon the change of variable

$$\rho = \sqrt{\frac{2\pi\rho'}{\kappa A_H}} \quad (126)$$

we get

$$d\tilde{s}_{NH}^2 = \frac{1}{\kappa^2} d\tilde{s}^2 \quad (127)$$

where

$$d\tilde{s}^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{4\rho^2} (d\theta^2 + \sin^2\theta d\phi^2). \quad (128)$$

Note that in these coordinates the brick-wall is situated at

$$\rho(\epsilon) = \sqrt{\frac{2\pi\epsilon}{\kappa A_H}} = \sqrt{\frac{\pi}{A_H}} \delta. \quad (129)$$

Also note that the scalar curvature of the optical metric  $d\tilde{s}_{NH}^2$  is

$$R = \kappa^2 \tilde{R}, \quad (130)$$

where  $\tilde{R}$  is the scalar curvature of  $d\tilde{s}^2$ . From now on we will denote geometric quantities derived from the metric  $d\tilde{s}^2$  with a tilde.

Similarly the Laplacian of the optical metric is related to the Laplacian of  $d\tilde{s}^2$  as

$$\Delta = \kappa^2 \tilde{\Delta}. \quad (131)$$

$$\begin{aligned} U_1 &= \frac{1}{6}R + F \left[ m^2 + \left( \xi - \frac{1}{6} \right) \mathcal{R} \right] \\ &= \kappa^2 \tilde{U}_1, \end{aligned} \quad (132)$$

where

$$\tilde{U}_1 = \frac{1}{6}\tilde{R} + \frac{F}{\kappa^2} \left[ m^2 + \left( \xi - \frac{1}{6} \right) \frac{8\pi}{A_H} \right] \quad (133)$$

But since

$$\frac{F}{\kappa^2} = \frac{2\kappa\rho'}{\kappa^2} = \frac{A_H}{\pi}\rho^2 \quad (134)$$

is independent of  $\kappa$  we see that  $H_1$  scales like  $\kappa^2$ ,

$$H_1 = \kappa^2 \tilde{H}_1, \quad (135)$$

where

$$\tilde{H}_1 = -\tilde{\Delta} + \tilde{U}_1. \quad (136)$$

Finally for some dimensionless parameter  $r$  (which will play the role of dimensionfull  $m'$  of Sec. 2) we define

$$\tilde{U} = \tilde{U}_1 - r, \quad (137)$$

and

$$\tilde{H} = -\tilde{\Delta} + \tilde{U}. \quad (138)$$

Thus

$$\tilde{H}_1 = \tilde{H} + r, \quad (139)$$

and

$$H_1 = \kappa^2 \tilde{H} + r\kappa^2. \quad (140)$$



So we get

$$\begin{aligned}\sum_{\sigma} e^{-k\beta\epsilon_{\sigma}} &= \sum_{\sigma} e^{-(k\beta r)(r^{-1}\epsilon_{\sigma})} \\ &= \frac{k\beta r}{2\sqrt{\pi}} \int_0^{\infty} \frac{dv}{v^{3/2}} e^{-\frac{(k\beta r)^2}{4v}} e^{-v\kappa^2} Tr e^{-v\kappa^2 r^{-1}\tilde{H}}\end{aligned}\quad (141)$$

Upon the change of variable  $u = v\kappa^2$  we get

$$\sum_{\sigma} e^{-k\beta\epsilon_{\sigma}} = \frac{k\beta r\kappa}{2\sqrt{\pi}} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{(k\beta r\kappa)^2}{4u}} e^{-u} Tr e^{-ur^{-1}\tilde{H}}. \quad (142)$$

So the free energy is

$$\begin{aligned}\mathcal{F} &= -\sum_{k=1}^{\infty} \frac{1}{k\beta} \sum_{\sigma} e^{-k\beta\epsilon_{\sigma}} \\ &= -\frac{\kappa r}{2\sqrt{\pi}} \sum_{k=1}^{\infty} \int_0^{\infty} \frac{du}{u^{3/2}} e^{-\frac{(k\beta r\kappa)^2}{4u}} e^{-u} Tr e^{-ur^{-1}\tilde{H}}.\end{aligned}\quad (143)$$

Comparing this with (20) we see that it can be obtained from the latter by performing the substitutions

$$\beta \rightarrow \beta\kappa, \quad m' \rightarrow r \quad (144)$$

in the latter and by multiplying the result by  $\kappa$ . The high temperature expansion is obtained by performing the same substitutions and multiplication in (51). Note that in the resulting expression  $T$  appears only through the combination  $T/\kappa$ . Finally for the high temperature expansion of the entropy we get

$$\begin{aligned}S &= -\frac{\partial(\mathcal{F}/\kappa)}{\partial(T/\kappa)} = 4\zeta(4)\frac{1}{\pi^2} \tilde{V}(T\kappa^{-1})^3 - 3\zeta(3) \left(\frac{1}{8\pi} \tilde{A}\right) (T\kappa^{-1})^2 \\ &+ \frac{\zeta(2)}{2\pi^2} \left[ \int_B d\tilde{V} \frac{(-A_H\rho^2)}{\pi} \left[ m^2 + \left(\xi - \frac{1}{6}\right) \mathcal{R} \right] + \int_{\partial B} dS \frac{\tilde{K}_{aa}}{3} \right] (T\kappa^{-1}) \\ &+ \frac{1}{16\pi} \int_{\partial B} dS \left\{ \frac{-A_H\rho^2}{\pi} \left[ m^2 + \left(\xi - \frac{1}{6}\right) \mathcal{R} \right] + \right. \\ &+ \frac{1}{96} [8\tilde{R}_{NaNa} + 7\tilde{K}_{aa}\tilde{K}_{bb} - 10\tilde{K}_{ab}\tilde{K}_{ab}] \Big\} \log(T\kappa^{-1}r^{-1}) \\ &+ \dots\end{aligned}\quad (145)$$

First note that at  $T = T_H$ ,  $T\kappa^{-1} = 2/\pi$  and therefore all the terms in the high temperature expansion become of the same order of magnitude. Second since the metric (128) does not involve any dimensionfull parameters the curvature terms in the heat kernel expansion will not involve such parameters either. The only exception to this is the potential term (133).

Now we have

$$\tilde{V} = 4\pi \int_{\rho(\epsilon)}^L \frac{d\rho}{\rho^3} \asymp 4\pi \frac{1}{2\rho^2(\epsilon)} = \frac{A_H}{2\delta^2}, \quad (146)$$

and similarly,

$$\tilde{A} \asymp \frac{1}{\rho^2(\epsilon)} = \frac{A_H}{2\delta^2}. \quad (147)$$

If choose the following orthonormal basis adapted to  $\rho = \text{const.}$  surfaces

$$N = \rho\partial_\rho, \quad E_1 = \rho\partial_\theta, \quad E_2 = \frac{\rho}{\sin\theta}\partial_\phi. \quad (148)$$

Then from (66) the extrinsic curvature of a  $\rho = \text{const.}$  surface is calculated to be

$$\tilde{K}_{ab} = \delta_{ab}. \quad (149)$$

We also have

$$\tilde{R} = -6 + 2\rho^2, \quad (150)$$

and

$$\tilde{R}_{aNaN} = -2. \quad (151)$$

After performing the similar algebra as in the previous cases, we arrive at

$$\begin{aligned} S = & \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} - \frac{3}{8}\zeta(3) + \frac{2}{3}\zeta(2) - \frac{\pi^2}{48} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ & - \frac{\zeta(2)}{8\pi^3} \left[ m^2 A_H + \left( \xi - \frac{1}{6} \right) (A_H \mathcal{R}) \right] \log \frac{A_H}{\pi\delta^2}. \end{aligned} \quad (152)$$

Now note that choosing  $r = 1/2\pi$  we can eliminate the logarithmic term in the  $\delta^{-2}$  divergence. The coefficient of  $\delta^{-2}$  term is clearly the same  $B$  coefficient we derived in earlier sections. Thus, in accordance with [31], we conclude that the common near horizon geometry of the backgrounds we considered is the basic reason for the equality of their  $B$  coefficients.

On the other hand the coefficient of the logarithmic divergence has a missing piece, namely the  $\zeta(4)$  term coming from the volume of the spherical shell. Most probably this situation can be remedied by going to sub-leading

orders in the near horizon approximation. A more important observation is that the logarithmic term generates a finite piece proportional to the logarithm of the horizon area. Using  $\mathcal{R} = 8\pi/A_H$  and  $\zeta(2) = \pi^2/6$ , we see that this finite piece is

$$\frac{1}{6} \left[ \left( \frac{1}{6} - \xi \right) - \frac{m^2 A_H}{8\pi} \right] \log A_H. \quad (153)$$

A detailed discussion of this term and its comparison with the existing results (e.g. [47, 48]) about logarithmic corrections to black hole entropy will be given in a future work.

## 6 Thermodynamics in the Presence of Neumann Walls

In this section we will consider the effects of Neumann boundary conditions on the entropy. Since we impose Neumann boundary conditions on the original mode functions  $f_\sigma$  the rescaled mode functions  $\bar{f}_\sigma$  will satisfy generalized Neumann boundary conditions

$$B\phi = (\phi_{;N} + J\phi) |_{\partial M} = 0 \quad (154)$$

We calculate  $J$  with a little algebra;

$$\bar{f}_\sigma = F^{\frac{1}{2}} f_\sigma, \quad f_{\sigma;N} |_{\partial M} = 0 \quad (155)$$

$$\bar{f}_{\sigma;N} = \left( \frac{1}{2} F^{-1/2} F_{;N} \right) F^{-1/2} \bar{f}_\sigma, \quad \bar{f}_{\sigma;N} - \frac{1}{2} \frac{F_{;N}}{F} \bar{f}_\sigma = 0 \quad (156)$$

$$J = -\frac{1}{2} \frac{F_{;N}}{F}. \quad (157)$$

For the Neumann problem first few heat kernel coefficients are given as

$$\begin{aligned} a_0 &= \frac{1}{(4\pi)^{d/2}} \int_B dV, \\ a_{1/2} &= \frac{1}{4(4\pi)^{\frac{d-1}{2}}} \int_{\partial B} dS, \\ a_1 &= \frac{1}{6(4\pi)^{d/2}} \left[ \int_B dV (-6U + R) + 2 \int_{\partial B} dS (K_{aa} + 6J) \right], \\ a_{3/2} &= \frac{1}{4} \frac{1}{(4\pi)^{(d-1)/2}} \frac{1}{96} \int_{\partial B} dS [16(-6U + R) + 8R_{aNaN} + \\ &\quad + 13K_{aa}K_{bb} + 2K_{ab}K_{ab} + 96JK_{aa} + 192J^2]. \end{aligned} \quad (158)$$

Comparing these coefficients with the ones for the Dirichlet problem (39) we see here some extra  $J$  terms and some sign changes. With these minor changes the calculations proceed exactly as in the Dirichlet case. Here we shall report only the final results for the entropy.

For the Schwarzschild Black Hole we obtain

$$\begin{aligned} S &= \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8}\zeta(3) - \frac{4}{3}\zeta(2) - \frac{115\pi^2}{192} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}. \end{aligned} \quad (159)$$

For the Reissner-Nordst rm Black Hole we get

$$\begin{aligned} S &= \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8}\zeta(3) - \frac{4}{3}\zeta(2) - \frac{115\pi^2}{192} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{2r_+ - 3r_-}{2r_+} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \left( \frac{\delta^2}{L^2} \right). \end{aligned} \quad (160)$$

and for the extreme case;

$$\begin{aligned} S &= \frac{A_H}{4\pi^3} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8}\zeta(3) - \frac{4}{3}\zeta(2) - \frac{115\pi^2}{192} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ \frac{\zeta(4)}{\pi^2} \frac{1}{2} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \left( \frac{\delta^2}{L^2} \right) \\ &+ \dots \end{aligned} \quad (161)$$

Finally for the Dilaton Black Hole;

$$\begin{aligned} S &= \frac{A_H}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8}\zeta(3) - \frac{4}{3}\zeta(2) - \frac{115\pi^2}{192} \log \frac{\kappa}{2\pi m'} \right] \frac{1}{\delta^2} \\ &+ \frac{1}{\pi^2} \left[ -\frac{\zeta(4)}{\pi^2} \frac{(8M - 3a)}{8M} + \frac{\zeta(2)}{8\pi} m^2 A_H \right] \log \frac{\delta^2}{8ML}. \end{aligned} \quad (162)$$

Thus for the Neumann boundary conditions we get

$$B = \frac{1}{(4\pi^3)} \left[ \frac{\zeta(4)}{\pi^2} + \frac{3}{8}\zeta(3) - \frac{4}{3}\zeta(2) \right], \quad (163)$$

$$C(\kappa, A_H, a) = \frac{\zeta(4)}{2\pi^4} \left[ \frac{1}{2} - \frac{3}{2\sqrt{\pi}} \kappa A_H \frac{1 - a\kappa}{\sqrt{1 - 2a\kappa}} \right] + \frac{\zeta(2)}{8\pi^3} m^2 A_H. \quad (164)$$

Unfortunately for Neumann walls  $B$  turns out to be negative, however this most probably means that one has to include more terms in the high temperature expansion to get a positive  $B$ .

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